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A Super-Soliton Hierarchy and Its Super-Hamiltonian Structure

Zhu Li · Huanhe Dong · Hongwei Yang

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Abstract A super-soliton hierarchy and its super-Hamiltonian structure is obtained respectively based on Lie super-algebra and associated super-trace identity.

Keywords Lie superalgebra · Supertrace identity · Superintegrable system · Super-Hamiltonian structure

1 Introduction

A simple and efficient method to obtain continuous or discrete integrable systems was proposed by Gui-zhang Tu in [1, 2]. Wen-xiu Ma further developed it and called it Tu model [3]. By taking advantage of it a family of integrable systems associated with physics backgrounds have been obtained, such as AKNS hierarchy, KN hierarchy, BPT hierarchy, etc. in [1–12]. With the development of soliton theory, recently, Wen-xiu Ma proposed a method to obtain super-integrable system in [13]. The main ideas are as follows:

Let $\mathcal{A}$ be a commutative superalgebra over R or C, and G a matrix loop superalgebra over $\mathcal{A}$ with the nondegenerate Killing form. Based on G we consider the following isospectral problems

$$\varphi_x = U\varphi = U(u,\lambda), \qquad \varphi_t = V\varphi, \qquad \lambda_t = 0, \tag{1}$$

where $u = (u_1, u_2, \ldots, u_q)^T \in \mathcal{A}^q$ is a potential consisting of commuting and anticommuting variables, λ is a spectral parameter.

The compatibility of (1) is the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{2}$$

where $[U, V] = UV - VU$.

Z. Li (✉)
College of Mathematics and Information Science, Xinyang Normal University, Xinyang, 464000, China
e-mail: lizhu1813@163.com

H. Dong · H. Yang
College of Information Science and Engineering, Shandong University of Science and Technology, Qingdao, 266510, China

If a equation

$$u_t = K(u) \tag{3}$$

can be work out through (2), we call (3) is a super-evolution equation,

If there is a super-Hamiltonian operator J and a functional $\mathcal{H}$ such that

$$u_t = K(u) = J\frac{\delta H}{\delta u}, \tag{4}$$

then (3) is called a super-Hamiltonian equation. If so, we say that (3) has a super-Hamiltonian structure.

2 The Super-Soliton Hierarchy

We first construct the following Lie superalgebra G

$$\begin{cases} e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ e_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad [e_1, e_2] = e_3, \quad [e_1, e_3] = e_2, \\ [e_3, e_2] = e_1, \quad [e_1, e_4] = [e_2, e_5] = [e_3, e_5] = \frac{e_4}{2}, \quad [e_5, e_1] = [e_2, e_4] = [e_4, e_3] = \frac{e_5}{2}, \\ [e_4, e_5]_+ = [e_5, e_4]_+ = \frac{e_1}{2}, \quad [e_4, e_4]_+ = -\frac{e_2+e_3}{2}, \quad [e_5, e_5]_+ = \frac{e_2-e_3}{2}, \end{cases} \tag{5}$$

where e_1, e_2, e_3, are even and e_4, e_5 are odd, and $[\cdot,\cdot]$ and $[\cdot,\cdot]_+$ denote the commutator and the anticommutator. The corresponding loop superalgebra $\tilde{G}$ is given as follows

$$\begin{cases} e_1 = \frac{1}{2}\lambda^n\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \frac{1}{2}\lambda^n\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2}\lambda^n\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ e_4 = \frac{1}{2}\lambda^n\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_5 = \frac{1}{2}\lambda^n\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad [e_1(m), e_2(n)] = e_3(m+n), \\ [e_1(m), e_3(n)] = e_2(m+n), \quad [e_3(m), e_2(n)] = e_1(m+n), \\ [e_1(m), e_4(n)] = \frac{e_4(m+n)}{2}, \quad [e_2(m), e_5(n)] = \frac{e_4(m+n)}{2}, \quad [e_3(m), e_5(n)] = \frac{e_4(m+n)}{2}, \\ [e_5(m), e_1(n)] = \frac{e_5(m+n)}{2}, \quad [e_2(m), e_4(n)] = \frac{e_5(m+n)}{2}, \quad [e_4(m), e_3(n)] = \frac{e_5(m+n)}{2}, \\ [e_5(m), e_5(n)]_+ = \frac{e_2(m+n)-e_3(m+n)}{2}, \quad [e_4(m), e_4(n)]_+ = -\frac{e_2(m+n)+e_3(m+n)}{2}, \\ [e_4(m), e_5(n)]_+ = [e_5(m), e_4(n)]_+ = \frac{e_1}{2}(m+n). \end{cases} \tag{6}$$

Considering the super-isospectral problem as follows

$$\begin{aligned} &\varphi_x = [U, \varphi], \quad U = e_1(-1) + u_1 e_2(0) + u_2 e_3(0) + u_3 e_4(0) + u_4 e_5(0), \\ &\lambda_t = 0. \end{aligned} \tag{7}$$

Taking

$$V = \sum_{m=0}^{\infty}(a_m e_1(m) + b_m e_2(m) + c_m e_3(m) + d_m e_4(m) + f_m e_5(m)). \tag{8}$$

Solving the stationary zero curvature equation $V_x = [U, V]$, give rise to

$$\begin{cases} a_{mx} = u_2 b_m - u_1 c_m + \frac{1}{2} u_4 d_m + \frac{1}{2} u_3 f_m, \\ b_{mx} = c_{m+1} - u_2 a_m - \frac{1}{2} u_3 d_m + \frac{1}{2} u_4 f_m, \\ c_{mx} = b_{m+1} - u_1 a_m - \frac{1}{2} u_3 d_m - \frac{1}{2} u_4 f_m, \\ d_{mx} = \frac{1}{2} d_{m+1} + \frac{1}{2} u_1 f_m + \frac{1}{2} u_2 f_m - \frac{1}{2} u_3 a_m - \frac{1}{2} u_4 b_m - \frac{1}{2} u_4 c_m, \\ f_{mx} = -\frac{1}{2} f_{m+1} + \frac{1}{2} u_1 d_m - \frac{1}{2} u_2 d_m - \frac{1}{2} u_3 b_m + \frac{1}{2} u_3 c_m + \frac{1}{2} u_4 a_m, \\ a_0 = \alpha = \text{const} \neq 0, \quad b_0 = c_0 = d_0 = f_0 = 0, \quad b_1 = \alpha u_1, \\ a_1 = \alpha \partial^{-1} u_3 u_4, \quad c_1 = \alpha u_2, \quad d_1 = \alpha u_3, \quad f_1 = \alpha u_4. \end{cases} \tag{9}$$

Denoting

$$\begin{aligned} V_-^{(n)} &= \sum_{m=0}^{n} (a_m e_1(m-n) + b_m e_2(m-n) + c_m e_3(m-n) \\ &\quad + d_m e_4(m-n) + f_m e_5(m-n)), \\ V_+^{(n)} &= \lambda^{-n} V - V_-^{(n)}. \end{aligned} \tag{10}$$

A direct calculation reads

$$-V_{-x}^{(n)} + [U, V_-^{(n)}] = c_{n+1} e_2(0) + b_{n+1} e_3(0) + \frac{1}{2} d_{n+1} e_4(0) - \frac{1}{2} f_{n+1} e_5(0). \tag{11}$$

Taking $V^{(n)} = V_-^{(n)}$, then the zero curvature equation

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{12}$$

admits the following superintegrable system

$$u_t = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t = \begin{pmatrix} c_{n+1} \\ b_{n+1} \\ \frac{1}{2} d_{n+1} \\ -\frac{1}{2} f_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} b_{n+1} \\ -c_{n+1} \\ -f_{n+1} \\ d_{n+1} \end{pmatrix} = J P_{n+1}. \tag{13}$$

From the recursion relations in (9), a recurrence operation L meets $P_{n+1} = L P_n$, where

$$L = \begin{pmatrix} u_1 \partial^{-1} u_2 & u_1 \partial^{-1} u_1 - \partial & -\frac{1}{2} u_4 - \frac{1}{2} u_1 \partial^{-1} u_3 & \frac{1}{2} u_3 + \frac{1}{2} u_1 \partial^{-1} u_4 \\ -\partial - u_2 \partial^{-1} u_2 & -u_2 \partial^{-1} u_1 & -\frac{1}{2} u_4 + \frac{1}{2} u_2 \partial^{-1} u_3 & -\frac{1}{2} u_3 - \frac{1}{2} u_2 \partial^{-1} u_4 \\ u_3 - u_4 \partial^{-1} u_2 & u_3 - u_4 \partial^{-1} u_1 & -2\partial + \frac{1}{2} u_4 \partial^{-1} u_3 & u_2 - u_1 - \frac{1}{2} u_4 \partial^{-1} u_4 \\ u_4 + u_3 \partial^{-1} u_2 & -u_3 + u_3 \partial^{-1} u_1 & u_2 + u_1 - \frac{1}{2} u_3 \partial^{-1} u_3 & 2\partial + \frac{1}{2} u_3 \partial^{-1} u_4 \end{pmatrix}. \tag{14}$$

3 Super-Hamiltonian Structure of the System (13)

Let a spectral matrix U be defined by

$$U = U(u, \lambda) = e_0(\lambda) + u_1 e_1(\lambda) + \cdots + u_q e_q(\lambda), \quad u_i \in \mathcal{A}, E_i \in G, 1 \le i \le q, \tag{15}$$

where $\mathcal{A}$ is a commutative superalgebra over R or C, G is a matrix loop superalgebra over $\mathcal{A}$ with the nondegenerate Killing form, and $E_i \in G$, are $\mathcal{A}$ linearly independent. If we define $\text{rank}(U) = \text{rank}(\frac{\partial}{\partial x}) = \text{const}$. Assume that if two solutions $V_1, V_2 \in G$ of the stationary zero curvature equation $V_x = [U, V]$ possess the same rank, then they are $\mathcal{A}$ linearly dependent of each other: $V_1 = \gamma V_2$, $\gamma = \text{const}$. From [13] we have the following two theorem

Theorem 1 (The supertrace identity) *Let $U = U(u, \lambda) \in G$ be homogeneous in rank. Assume that the stationary zero curvature equation has a unique solution $V \in G$ of a fixed rank up to a constant multiplier. Then, there is a constant γ such that*

$$\frac{\delta}{\delta u}\int \text{str}(ad_V ad_{U_\lambda})dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\,\text{str}(ad_V ad_{\partial U/\partial u}) \tag{16}$$

holds for any solution $V \in G$ of stationary zero curvature equation, being homogeneous in rank.

Theorem 2 *Let V be a solution to the stationary zero curvature equation. Then the constant in the supertrace identity is given by*

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln|\text{str}(ad_V ad_V)|, \tag{17}$$

if $\text{str}(ad_V ad_V) \neq 0$.

Based on Lie superalgebra G in (5) and associated corresponding loop superalgebra $\tilde{G}$, a direct calculation gives

$$ad_a = \begin{pmatrix} 0 & a_3 & -a_2 & \frac{a_5}{2} & \frac{a_4}{2} \\ -a_3 & 0 & a_1 & -\frac{a_4}{2} & \frac{a_5}{2} \\ -a_2 & a_1 & 0 & -\frac{a_4}{2} & -\frac{a_5}{2} \\ -\frac{a_4}{2} & -\frac{a_5}{2} & -\frac{a_5}{2} & \frac{a_1}{2} & \frac{a_2+a_3}{2} \\ \frac{a_5}{2} & -\frac{a_4}{2} & \frac{a_4}{2} & \frac{a_2-a_3}{2} & -\frac{a_1}{2} \end{pmatrix} \tag{18}$$

for $a = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 \in \tilde{G}$, where $ad_a b = [a, b], a, b \in G$, the bracket $[\cdot, \cdot]$ is the Lie superbracket of G. So, if we define the supertrace as follows

$$\begin{aligned} &\text{str}(c) = c_{11} + c_{22} - c_{33}, \quad c = ab,\ a, b \in \tilde{G}, \\ &\text{str}(P) = p_{11} + p_{22} + p_{33} - p_{44} - p_{55}, \end{aligned} \tag{19}$$

where $c = (c_{ij})_{3\times3}$, $P = (p_{ij})_{5\times5}$ and ab is the matrix product of a and b, then we have

$$\text{str}(ad_a ad_b) = 3\text{str}(ab). \tag{20}$$

It is easy to compute that

$$\begin{aligned} &\text{str}(ad_V ad_{U_\lambda}) = -\frac{a}{2\lambda^2}, \\ &\text{str}(ad_V ad_{\partial U/\partial u_1}) = \frac{3}{2}b, \qquad \text{str}(ad_V ad_{\partial U/\partial u_2}) = -\frac{3}{2}c, \end{aligned}$$

$$\mathrm{str}(ad_V ad_{\partial U/\partial u_3}) = -\frac{3}{2}f, \qquad \mathrm{str}(ad_V ad_{\partial U/\partial u_4}) = \frac{3}{2}d. \tag{21}$$

According to the supertrace identity (16), we have

$$\frac{\delta}{\delta u}\int\left(-\frac{a}{2\lambda^2}\right)dx = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left(\frac{3}{2}b, -\frac{3}{2}c, -\frac{3}{2}f, \frac{3}{2}d\right)^T. \tag{22}$$

Comparing the coefficient of λ^n yields

$$\frac{\delta}{\delta u}\int\left(-\frac{1}{3}a_{n+2}\right)dx = (\gamma + n + 1)\,(b_{n+1}, -c_{n+1}, -f_{n+1}, d_{n+1})^T. \tag{23}$$

Since $\mathrm{str}(ad_V ad_V) = \frac{1}{2}\alpha^2 \neq 0$, we have $\gamma = 0$. Therefore,

$$P_{n+1} = \frac{\delta H_n}{\delta u}, \quad H_n = \int\left(-\frac{a_{n+2}}{3(n+1)}\right)dx, n \geq 0. \tag{24}$$

Hence, the system (13) has the following super-Hamiltonian structure

$$u_t = JP_{n+1} = J\frac{\delta H_n}{\delta u}, \quad H_n = \int\left(-\frac{a_{n+2}}{3(n+1)}\right)dx, n \geq 0. \tag{25}$$

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